Diffusion in Pulsating Flow in a Distensible Conduit

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An earlier study on interphase mass transfer in pulsating laminar flow has been extended to the case of a distensible tube. The physical situation studied corresponds to developed flow in a long tube with a traveling pressure wave of small amplitude impressed on the steady flow. The tube wall is free to expand radially. Asymptotic solutions are developed for large and small values of the frequency parameter. The interphase flux is much greater in distensible conduits than in the corresponding case in rigid conduits.

Interphase mass transfer in pulsating flow in a distensible conduit is of interest in studies on the cardiovascular system as well as in other studies involving a flexible phase barrier. For example, one might consider pulsatile flow as a method of improving mass transfer in the various devices which use semipermeable membranes.

In an earlier paper (*1*) an analytical solution to the problem of diffusion in pulsating flow in a rigid conduit was developed. Here the method of solution is extended to the case of pulsating flow in a distensible tube.

PULSATING FLOW IN A DISTENSIBLE CONDUIT

The flow under consideration was apparently first treated by Womersley (*4*). The basis for the model and the solution in a slightly more general form are outlined below.

Consider an incompressible Newtonian fluid flowing in a distensible long tube which is longitudinally constrained so that there is no motion of the tube wall in the axial direction. The tube wall is free to move radially but maintains a circular cross section at all times. A traveling wave of small amplitude is impressed on the steady flow and the pressure gradient is given by

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{1}{\rho}\left(\frac{dp}{dx}\right)_0 \left[1 + \lambda\, e^{i\beta\left(t - \frac{x}{C}\right)}\right] \tag{1}$$

where $(dp/dx)_0$ is the mean steady pressure gradient, β is the frequency, λ the amplitude, and C the complex wave velocity. In the special case of real C the expression represents a wave of constant amplitude. It is convenient to write C for the complex case as

$$C = \frac{C_0}{A - iB}$$

where C_0, the velocity of the wave in a nonviscous liquid, is determined in the usual way by the formula $C_0 = [(Eh)/(2R\rho_w)]^{1/2}$. This means that the phase velocity is C_0/A or, in other words, the wave length in space is $(2\pi\, C_0)/(A\beta)$. The wave decays exponentially with distance according to the factor $\exp(-(\beta Bx)/C_0)$. From a balance of forces on the wall of the constrained elastic tube, Womersley (*4*) derived the expression for C given below.

$$\frac{C_0}{C} = \sqrt{\frac{4(1-\sigma^2)}{4-\chi(\omega)}}$$

$$\chi(\omega) = -\frac{8i^{1/2}}{\omega}\,\frac{J_1\,(\omega i^{3/2})}{J_0\,(\omega i^{3/2})}$$

Poisson's ratio, σ, was taken to be 0.5. This relationship was employed in this study to account for the effect of damping. When the frequency parameter ω is very large, the phase of $(1/C)$ is almost zero and damping becomes negligible.

To simplify the flow problem, the following assumptions are adopted.

1. The wave velocity C_0 is large compared to the mean axial velocity U_0. C_0 is a function only of the physical properties and dimensions of the conduit, namely, its density, Young's modulus of elasticity, the thickness of the wall, and the mean radius of the tube. In the cardiovascular system, the ideal wave velocity is 400 to 500 cm./sec. in the thoracic aorta, 500 to 650 cm./sec. in the abdominal aorta, 800 to 1,000 cm./sec. in the femoral artery, and larger in the small arteries (*3*). The ratio U_0/C_0 is about 10^{-1} in the aorta, 10^{-2} in the femoral arteries, and smaller in the smaller arteries.
2. The wave length is very much longer than the radius of the tube. For a normal individual the pulse rate is about 72 beats/min. Based on this frequency the average wave length is about 400 cm. (for $C_0 = 500$ cm./sec.) which

is more than 650 times larger than the radius of the aorta.

3. There are no reflected waves. In major arterial branches with an area ratio of 1.15 the reflection coefficient (that is, the amplitude of the reflected wave expressed as a fraction of the incident wave) for $\omega = [(\beta R^2/\nu)]^{1/2} = 5$ is about 4% and for $\omega = 10$, about 2% (3). The reflection coefficient for small vessels of 1 mm. diameter or less is slightly higher than these values.

By making use of the above three assumptions it can be shown (2) that the inertial terms are small compared to the acceleration term $\partial u/\partial t$. Here u is the velocity in the axial direction. Furthermore, the derivative in the axial direction, $\partial^2 u/\partial x^2$, is small compared to $1/r(\partial)/(\partial r)[r(\partial u/\partial r)]$. The differential equations for the simplified flow problem then become

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r} rv = 0 \tag{3}$$

The boundary conditions are

$$u = 0 \quad \text{at} \quad r = R + \alpha \quad \text{for all } x \text{ and } t \tag{4a}$$

$$\frac{\partial u}{\partial r} = 0 \quad \text{at} \quad r = 0 \quad \text{for all } x \text{ and } t \tag{4b}$$

$$v = \frac{\partial \alpha}{\partial t} \quad \text{at} \quad r = R + \alpha \quad \text{for all } x \text{ and } t \tag{4c}$$

In these expressions, α is the radial expansion of the wall, R is the average radius of the tube. Equations (4a) and (4c) stand for the condition of no slip at the wall and (4b) for symmetry.

The solution of the flow problem can be shown to have the form

$$u(r, x, t) = u_0(r) + \lambda\, u_1(r)\, e^{i\beta\left(t - \frac{x}{C}\right)}$$
$$v(r, x, t) = v_0(r) + \lambda\, v_1(r)\, e^{i\beta\left(t - \frac{x}{C}\right)} \tag{5}$$

where the basic terms in the velocity expressions are given by the Poiseuille flow relations

$$u_0(r) = 2U_0\left(1 - \frac{r^2}{R^2}\right)$$
$$v_0(r) = 0$$

The differential equations for $u_1(r)$ and $v_1(r)$ obtained by substituting Equation (5) into Equations (2) and (3) and equating the coefficients of λ to zero, are given by

$$i\beta\, u_1 = -\frac{1}{\rho}\left(\frac{dp}{dx}\right)_0 + \frac{\nu}{r}\frac{d}{dr}\left(r\frac{du_1}{dr}\right) \tag{6a}$$

$$-\frac{i\beta\, u_1}{C} + \frac{1}{r}\frac{d}{dr} rv_1 = 0 \tag{6b}$$

To derive the boundary conditions for these equations it is necessary to express the radial expansion, α, in terms of the mean velocity.

$$\frac{\partial \alpha}{\partial t} = v\,(R + \alpha, x, t) = \lambda\, v_1\,(R + \alpha)\, e^{i\beta\left(t - \frac{x}{C}\right)}$$

The function $v_1\,(R + \alpha)$ is then expanded in a Taylor series about $r = R$ giving

$$\frac{\partial \alpha}{\partial t} = \lambda\, e^{i\beta\left(t - \frac{x}{C}\right)}\left[v_1(R) + \alpha\left(\frac{dv_1}{dr}\right)_{r=R} + O(\alpha^2)\right] \tag{7}$$

The value of v_1 at $r = R$ is obtained from the continuity relationship (6b).

$$v_1(R) = \frac{i\beta}{2C}\,\frac{\int_0^R 2\pi r u_1 dr}{\pi R^2} = \frac{i\,\beta\, R\,\overline{u}_1}{2C} \tag{8}$$

Integration of Equation (7) yields an expression for α.

$$\alpha \cong \frac{\lambda\, R\,\overline{u}_1}{2C}\, e^{i\beta\left(t - \frac{x}{C}\right)} \tag{9}$$

The condition of no tangential flow at the tube wall (4b) is rewritten in the form of Equation (5), giving

$$u_0(R + \alpha) + \lambda\, u_1(R + \alpha)\, e^{i\beta\left(t - \frac{x}{C}\right)} = 0$$

Here the parameter λ appears explicitly as well as implicitly in the arguments of the function u_0 and u_1. In this form, it is not possible directly to equate like powers of λ to zero. However, assuming that u_0 and u_1 are both analytic in their dependence on the variable r, both functions can be expanded in a Taylor series. Then making use of Equation (9), the nonslip condition at the wall is finally expressed as

$$u_0(R) + \lambda\, e^{i\beta\left(t - \frac{x}{C}\right)}\left[\frac{R\,\overline{u}_1}{2C}\left(\frac{du_0}{dr}\right)_{r=R} + u_1(R)\right] + O(\lambda^2) = 0 \tag{10}$$

Thus, the boundary conditions required to solve for u_1 are given by

$$\left(\frac{du_1}{dr}\right)_{r=0} = 0 \tag{11a}$$

$$u_1(R) + \frac{R\,\overline{u}_1}{2C}\left(\frac{du_0}{dr}\right)_{r=R} = 0 \tag{11b}$$

Equations (6a), (11a), and (11b) completely define the fluctuating component u_1.

$$u_1(R) = \frac{8U_0}{i\omega^2}\left[1 - a\,\frac{J_0\left(\omega \frac{r}{R}\, i^{3/2}\right)}{J_0(\omega i^{3/2})}\right] \tag{12}$$

Here the constant a is

$$a = \frac{1 - \dfrac{U_0}{C}}{1 - \dfrac{2\left(\dfrac{U_0}{C}\right) J_1(\omega i^{3/2})}{\omega i^{3/2}\, J_0(\omega i^{3/2})}}$$

Having found $u_1(r)$, Equation (6b) can now be employed to give

$$v_1(r) = \frac{8\nu}{R}\left(\frac{U_0}{C}\right)\left[\frac{r}{2R} - a\,\frac{J_1\left(\omega \frac{r}{R}\, i^{3/2}\right)}{J_0\,(\omega i^{3/2})}\right] \tag{13}$$

When U_0/C is small, the constant a is approximately equal to 1. This is the value found by Womersley (4). Womersley did not derive an expression for v. However, it can be shown that his work could be extended to yield a value of v which differs from the one presented here but which reduces to the one presented here as U_0/C vanishes.

CONCENTRATION DISTRIBUTION AND DIFFUSION

In solving the diffusion problem it is assumed that mass transport is of a boundary layer nature even in the region of fully developed flow. This is a valid approximation for large Schmidt numbers which is of course the case for liquids. The specific range of validity has been given previously (1). The linear approximation to the velocity profile near the wall can then be considered.

$$u \cong \frac{4\,U_0\,y}{R} + \lambda \frac{U_0\,y}{R} \chi(\omega)\, e^{i\beta\left(t - \frac{x}{C}\right)}$$

$$v \cong \frac{\lambda\nu}{R}\left(\frac{U_0}{C}\right)\left(1 + \frac{y}{R}\right)[\chi(\omega) - 4]\, e^{i\beta\left(t - \frac{x}{C}\right)} \tag{14}$$

In these expressions, y is the distance from the tube wall and $\chi(\omega)$ is a function of ω which has been previously defined.

The boundary layer equation for the dimensionless concentration variable $\phi = (c - c_w)/(c_0 - c_w)$ is

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} = D\frac{\partial^2 \phi}{\partial y^2} \tag{15}$$

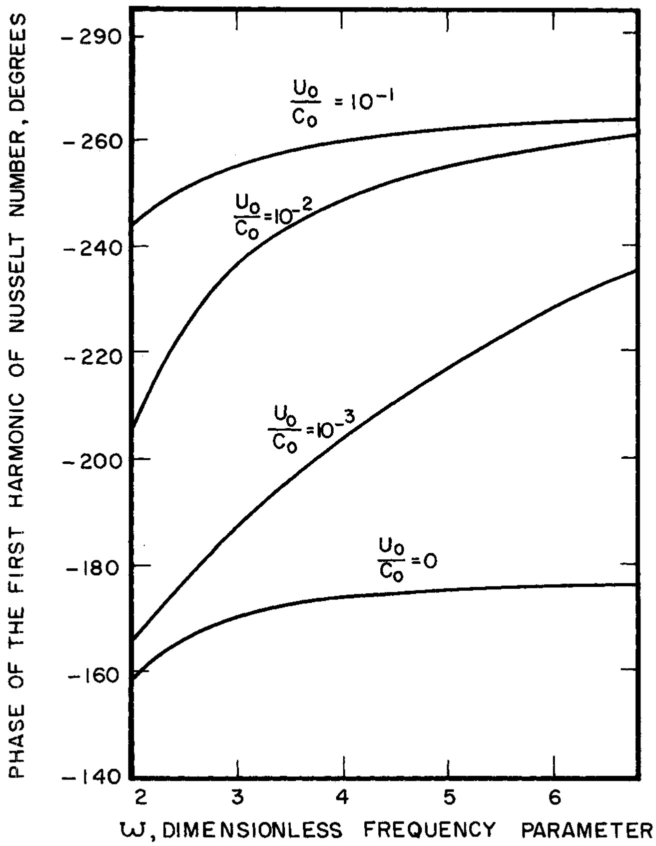

Fig. 1. Frequency dependence of the phase of the first harmonic of the Nusselt number. (Phase of $N_{Nu}^{(1)}$ for $\xi = 0.20 \times 10^{-2}$).

$$\begin{aligned} \phi &= 1 \quad \text{at} \quad x = 0 \quad \text{for all } t \text{ and } y \\ \phi &= 0 \quad \text{at} \quad y = 0 \quad \text{for all } t \text{ and } x > 0 \\ \phi &= 1 \quad \text{at} \quad y = \infty \quad \text{for all } t \text{ and } x > 0 \end{aligned} \tag{16}$$

Since the perturbation method of solution has already been described in detail elsewhere (1, 2), the account of the mathematical operations to be presented here will be brief. A second-order perturbation is sought in the amplitude parameter, λ.

$$\phi(x, y, t) = \phi_0(x, y) + \lambda\phi_1(x, y, t) + \lambda^2\phi_2(x, y, t) + O(\lambda^3) \tag{17}$$

The zeroth-order term of this expansion is the same as the steady state solution. The differential equation for the first and second-order terms differ from those of the rigid tube case only in the presence of an additional term involving the radial velocity. Solutions of these equations are then found in the high frequency and low frequency regions.

The solution of the first-order term is expressed as

$$\phi_1(x, y, t) = f_1(x, y)\, e^{i\beta\left(t - \frac{x}{C}\right)} \tag{18}$$

and the high frequency solution in terms of the derivative of $f_1(x, y)$ at the wall is

$$\left.\frac{\partial f_1}{\partial y}\right|_{y=0} = \frac{1}{Ai(\epsilon^{2/3})} \int_0^\infty Ai\left[\epsilon^{2/3}\left(1 - \frac{4Y}{R}\frac{U_0}{C}\right)\right] Q_1(x, y)\,dy \tag{19}$$

where

$$Q_1(x, y) = \frac{U_0\,y}{RD}\chi(\omega)\frac{\partial \phi_0}{\partial x} + \frac{N_{Sc}}{R}\left(\frac{U_0}{C}\right)\left(1 + \frac{y}{R}\right)[\chi(\omega) - 4]\frac{\partial \phi_0}{\partial y}$$

and

$$\epsilon = \frac{\bar{\omega} i^{1/2}}{4\frac{U_0}{C}}$$

In the limiting case of $|U_0/C| \to 0$ and $|\epsilon| \to \infty$, the asymptotic expansion of the above expression is given by

$$\left.\frac{\partial f_1}{\partial y}\right|_{y=0} \cong \frac{(12)^{1/3}}{R\Gamma(1/3)}\left\{ -\frac{\epsilon^{-2}}{\left(\frac{4U_0}{C}\right)^2} N_{Sc}\frac{U_0}{C}[4 - \chi(\omega)]\xi^{-1/3} + \frac{2}{3}\frac{\epsilon^{-3}}{\left(\frac{4U_0}{C}\right)^3}\chi(\omega)\xi^{-4/3} \right\} \tag{20}$$

where ξ is the dimensionless variable Dx/U_0R^2.

For very slow pulses the first harmonic of the flux approximates that of flow in a rigid tube.

$$\left.\frac{\partial f_1}{\partial y}\right|_{y=0} = \frac{(12)^{1/3}}{\Gamma(1/3)}\left(\frac{U_0}{RDx}\right)^{1/3} + O(\omega^2) \tag{21}$$

The second-order term, $\phi_2(x, y, t)$ is decomposed into a steady component and a transient component which is a periodic function of time.

$$\phi_2(x, y, t) = M(x, y) + f_2(x, y) e^{2i\beta\left(t - \frac{x}{C}\right)} \tag{22}$$

A complete description of the procedure in the derivation of these solutions is available (2). Only the derivatives of the functions $M(x, y)$ and $f_2(x, y)$ at the wall will be given here. The solutions in the high frequency region are

$$\left.\frac{\partial f_2}{\partial y}\right|_{y=0} = \frac{1}{Ai[(\sqrt{2})^{2/3}]} \int_0^\infty Ai\left[(\sqrt{2}\,\epsilon)^{2/3}\left(1 - \frac{U_0}{C}\frac{4y}{R}\right)\right] Q_2(x, y)\, dy \tag{23}$$

$$\left.\frac{\partial M}{\partial y}\right|_{y=0} = -\frac{(12)^{1/3}}{3\Gamma(1/3)\,\frac{U_0}{RD}} \int_0^\infty \int_0^x G(\tau, y) y \left[\frac{U_0}{RD(x-\tau)}\right]^{4/3} \exp\left[-\frac{4}{9} y^3 \frac{U_0}{RD(x-\tau)}\right] d\tau dy \tag{24}$$

where

$$Q_2(x, y) = \frac{U_0 y}{2RD}\chi(\omega)\frac{\partial f_1}{\partial x} + \frac{N_{Sc}}{2R}\frac{U_0}{C}\left(1 + \frac{y}{R}\right)[\chi(\omega) - 4]\frac{\partial f_1}{\partial y}$$

$$G(x, y) = \frac{U_0 y}{2RD}\left[Re[\chi(\omega)]Re\left(\frac{\partial f_1}{\partial x}\right) + Im[\chi(\omega)]\, Im\left(\frac{\partial f_1}{\partial x}\right)\right]$$

$$+ \frac{N_{Sc}}{2R}\left(1 + \frac{y}{R}\right)\left[Re\left\{\frac{U_0}{C}[\chi(\omega) - 4]\right\} Re\left(\frac{\partial f_1}{\partial y}\right) + Im\left\{\frac{U_0}{C}[\chi(\omega) - 4]\right\} Im\left(\frac{\partial f_1}{\partial y}\right)\right]$$

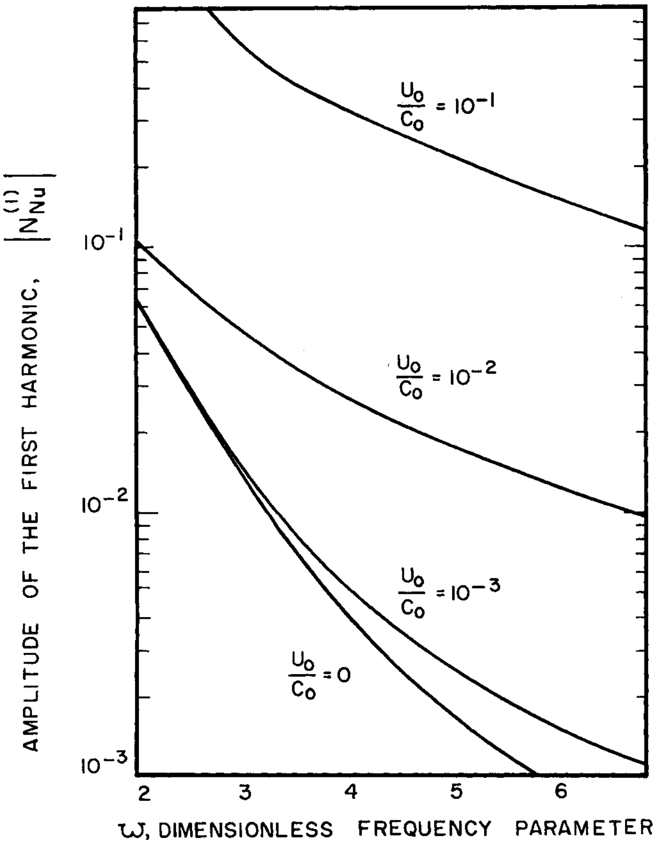

Fig. 2. Frequency dependence of the amplitude of the first harmonic of the Nusselt number. (Amplitude of $N_{Nu}^{(1)}$ for $\xi = 0.20 \times 10^{-2}$).

Approximate expressions may be obtained by replacing $f_1(x, y)$ in $Q_2(x, y)$ and $G(x, y)$ by its linear expansion about $y = 0$. This operation yields simpler expressions for the integrands and the integrals can then be evaluated in closed form. The dominant terms in the asymptotic expansion of the second-order terms for large $|\epsilon|$ and small $|U_0/C|$ are:

$$\left.\frac{\partial f_2}{\partial y}\right|_{y=0} \cong -\frac{(12)^{1/3}}{R\Gamma(1/3)} \left[-\frac{1}{4}\frac{\epsilon^{-4}}{4\left(\frac{4U_0}{C}\right)^4} N_{Sc}^2 \left(\frac{U_0}{C}\right)^2 [4 - \chi(\omega)]^2 \xi^{-1/3} + \frac{(2+\sqrt{2})}{6\sqrt{2}}\frac{\epsilon^{-5}}{\left(\frac{4U_0}{C}\right)^5} N_{Sc}\frac{U_0}{C}\chi(\omega)[4 - \chi(\omega)]\xi^{-4/3} - \frac{4}{9\sqrt{2}}\frac{\epsilon^{-6}}{\frac{4U_0}{C}}[\chi(\omega)]^2\xi^{-7/3}\right] \tag{25}$$

$$\left.\frac{\partial M}{\partial y}\right|_{y=0} \cong \frac{(12)^{1/3}}{R\Gamma(1/3)} \left\{\frac{1}{\bar{\omega}^2}\frac{N_{Sc}}{\omega}\frac{U_0}{C_0} A^{-1/2}\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right) \times \left[\left(\frac{2}{3}\right)^{1/6}\Gamma\left(\frac{2}{3}\right)\xi^{-2/3} + \left(\frac{3}{2}\right)^{1/2}\xi^{-1/3}\left[1 + \frac{2}{2^{1/3}}\frac{\sqrt{\pi}\,\Gamma(2/3)}{\Gamma(1/3)\,\Gamma(5/6)}\right]\right] + \frac{1}{\bar{\omega}^3}\left[\frac{16}{3\sqrt{2}}\frac{\xi^{-4/3}}{\omega^2} + 2\left(\frac{2}{3}\right)^{2/3} \times \frac{\sqrt{3\pi}\,\Gamma(2/3)}{\omega\Gamma(1/3)\,\Gamma(5/6)} N_{Sc}\frac{U_0}{C_0}A^{1/2}\xi^{-2/3}\cos\frac{\theta}{2} + \frac{3}{\sqrt{2}}\Gamma(2/3)\, N_{Sc}^2\left(\frac{U_0}{C_0}\right)^2 A\left((2/3)^{1/3}\xi^{-2/3} + \frac{2^{1/3}}{\Gamma(2/3)}\xi^{-1/3}\right)\right]\right\} \tag{26}$$

where $A = 4 - \chi(\omega)$ and $\theta = \arg.[4 - \chi(\omega)]$.

For very small frequencies, the solution in the limit is the same as that for flow in a rigid tube.

$$\left.\frac{\partial M}{\partial y}\right|_{y=0} = -\frac{(12)^{1/3}}{18\Gamma(1/3)}\left(\frac{U_0}{RDx}\right)^{1/3} + O(\omega)^2 \tag{27}$$

In many investigations, it is of utmost importance to be able to estimate the average mass transfer rate between two points along the wall. The dominant terms in the asymptotic expansion of the average Nusselt number for $|\epsilon| \to \infty$ and $|U_0/C| \to 0$ are written below.

$$\frac{F-1}{\lambda^2} \cong \frac{1}{\omega^3}\,\frac{U_0}{C_0}\,A^{1/2}\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)\left\{\frac{2\left(\frac{2}{3}\right)^{1/6}\Gamma\left(\frac{2}{3}\right)}{\xi_2^{1/3}\left[1+\left(\frac{\xi_1}{\xi_2}\right)^{1/3}\right]} + \left(\frac{2}{3}\right)^{1/2}\left[\frac{3}{2}+\frac{3\sqrt{\pi}\,\Gamma(2/3)}{2^{1/3}\,\Gamma(1/3)\,\Gamma(5/6)}\right]\right\}$$

$$+\frac{32\,N_{Sc}^{-3/2}}{3\sqrt{2}\;\omega^5}\;\frac{\left[1-\left(\frac{\xi_1}{\xi_2}\right)^{1/3}\right]}{\xi_1^{1/3}\,\xi_2^{2/3}\left[1-\left(\frac{\xi_1}{\xi_2}\right)^{2/3}\right]}$$

$$+\frac{4\left(\frac{2}{3}\right)^{2/3}\sqrt{3\pi}\,\Gamma(2/3)\,N_{Sc}^{-1/2}\,\frac{U_0}{C_0}\,A^{1/2}\cos\frac{\theta}{2}}{\Gamma(7/6)\,\Gamma(1/3)\;\omega^4\;\xi^{1/3}\left[1+\left(\frac{\xi_1}{\xi_2}\right)^{1/3}\right]}$$

$$+\frac{6}{\sqrt{2}}\,\Gamma(2/3)\;N_{Sc}^{1/2}\left(\frac{U_0}{C_0}\right)^2 A\left\{\frac{(2)^{1/3}}{2\Gamma(2/3)} + \frac{(2/3)^{1/3}}{\xi_2^{1/3}\left[1+\left(\frac{\xi_2}{\xi_1}\right)^{1/3}\right]}\right\} \qquad (28)$$

Here, $0 < \xi_1 < \xi_2$, and F is the ratio of the space and time-averaged Nusselt number to the Nusselt number for steady flow.

RESULTS

The Nusselt number for this diffusion problem is given by

$$N_{Nu} = N_{Nu}^{(0)} + \lambda\, N_{Nu}^{(1)} + \lambda^2\left[\overline{N_{Nu}^{(2)}} + N_{Nu}^{(2)}(t)\right] \qquad (29)$$

$$= 2R\,\frac{\partial \phi_0}{\partial y}\bigg|_{y=0} + 2R\lambda\,\frac{\partial f_1}{\partial y}\bigg|_{y=0} e^{i\beta\left(t-\frac{x}{C}\right)} + 2R\lambda^2\left[\frac{\partial f_2}{\partial y}\bigg|_{y=0} e^{2i\beta\left(t-\frac{x}{C}\right)} + \frac{\partial M}{\partial y}\bigg|_{y=0}\right]$$

in which $N_{Nu}^{(0)}$ represents the solution for steady flow, $N_{Nu}^{(1)}$ and $N_{Nu}^{(2)}(t)$ are periodic functions of time with zero time average and are designated as the first and second harmonics, respectively. $\overline{N_{Nu}^{(2)}}$ is independent of time and can be thought of as representing the net change in the time-averaged flux due to the pulsations.

Figures 1 to 5 give the high frequency solution for the Nusselt number. The solutions were obtained by integrating Equations (19), (23), and (24) using Gauss' numerical quadrature method. The phase lead (referred to a pure cosine pulse wave) and the amplitudes of $N_{Nu}^{(1)}$ as a function of ω are shown in Figures 1 and 2. The curves for the rigid flow case ($U_0/C_0 = 0$) are included for comparison. The asymptotic value of the phase of the first harmonic of the Nusselt number at infinite frequency is $(-3\pi/2)$ for the distensible tube case compared to $(-\pi)$ for flow in a

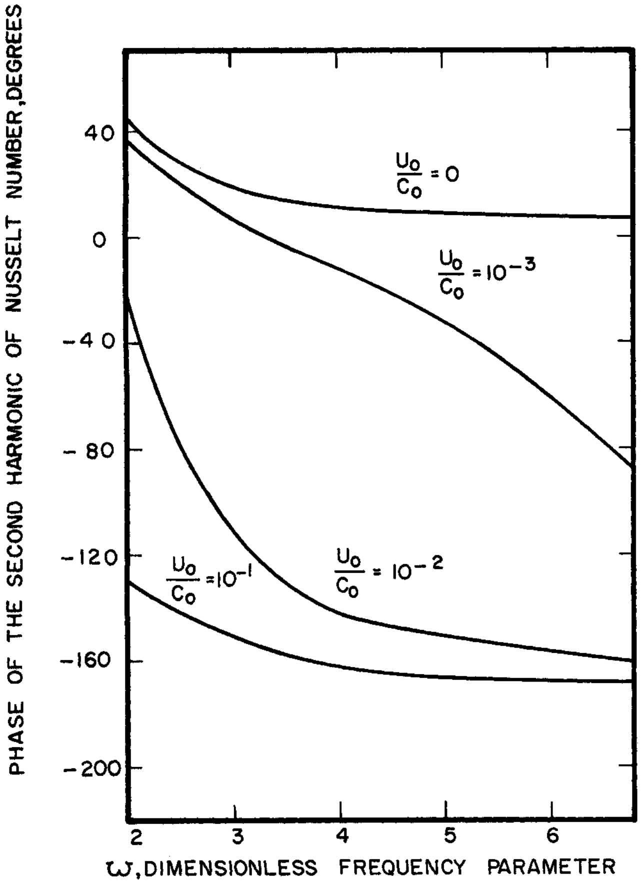

Fig. 3. Frequency dependence of the phase of the second harmonic of the Nusselt number. (Phase of $N_{Nu}^{(2)}(t)$ for $\xi = 0.20 \times 10^{-2}$).

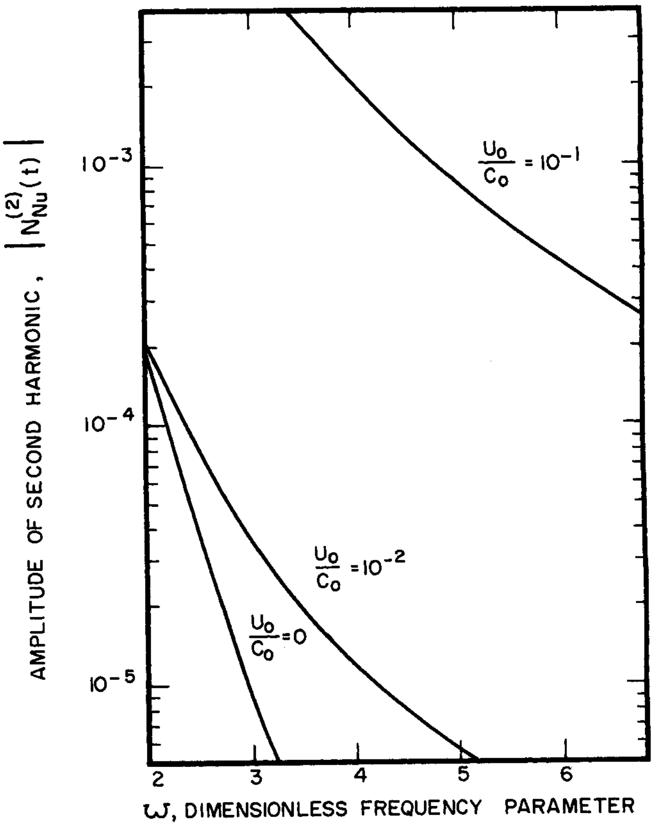

Fig. 4. Frequency dependence of the amplitude of the second harmonic of the Nusselt number. (Amplitude of $N_{Nu}^{(2)}(t)$ for $\xi = 0.20 \times 10^{-2}$.

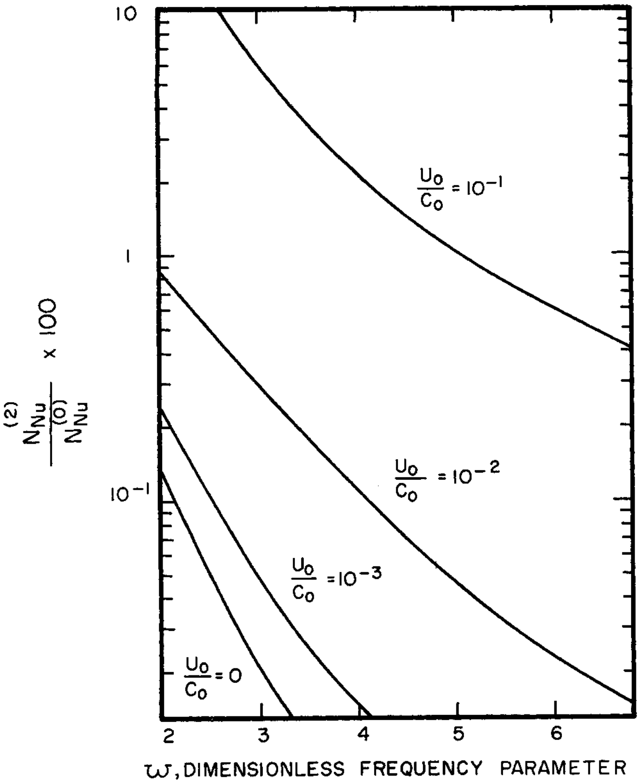

Fig. 5. Frequency dependence of the increase in the time-averaged Nusselt number over the steady flow value.

rigid tube. The amplitude of $N_{Nu}^{(1)}$ for a given pulse frequency increases as the elasticity of the tube becomes greater. These results can be verified by looking at the asymptotic expansion of $\partial f_1/\partial y|_{y=0}$, Equation (20). It can also be deduced from this equation that the dominant term of $N_{Nu}^{(1)}$ varies inversely as the square of ω for nonzero values of U_0/C_0 whereas for U_0/C_0 equal to zero, $N_{Nu}^{(1)}$ is of the order $1/\omega^4$.

Figures 3 and 4 show the frequency dependence of the phase lead (again referred to a pure cosine pulse wave of the same frequency) and the amplitude of $N_{Nu}^{(2)}(t)$. The phase of the second harmonic for flow in a rigid tube approaches zero as the frequency becomes infinite. In contrast, the asymptotic value of the phase lead for flow in a distensible conduit is $(-\pi)$. The behavior of the amplitude of $N_{Nu}^{(2)}(t)$ with respect to a variation of U_0/C_0 for fixed frequency is similar to that of $N_{Nu}^{(1)}$. However, the amplitube of $N_{Nu}^{(2)}(t)$ is less than that of $N_{Nu}^{(1)}$ by at least three orders of magnitude. Again, these features are clearly brought out by the asymptotic expansion of $\partial f_2/\partial y|_{y=0}$. In the case of flow in a distensible conduit $N_{Nu}^{(2)}$ is noted to be inversely proportional to ω^4 while for a rigid tube the amplitude of $N_{Nu}^{(2)}$ varies inversely to ω^8.

The time averaged Nusselt number which gives the net interphase flux, is a sum of the zeroth-order term and the steady component of the second-order term. The increase in the flux as a result of the pulsation in flow is a function of the amplitude parameter λ, the frequency parameter ω, the axial position ξ, and the ratio U_0/C_0. The variation of the change in flux with respect to ω at $\xi = 0.20 \times 10^{-2}$ for several values of U_0/C_0 is shown in Figure 5. It can be observed that the flux increases with U_0/C_0. This finding has significant implications. For example, when the arterial wall has become stiff and hardened by disease, U_0/C_0 decreases, and a diminution in the flux to the wall results. It can also be seen from Figure 5 that $\overline{N_{Nu}^{(2)}}$ takes on lower values as ω becomes greater for all values of U_0/C_0. However, this decrease is more sharply pronounced for flow in a rigid tube. This result is borne out by the asymptotic expansion of $\partial M/\partial y|_{y=0}$ which shows the time-independent second-order term of the flux to be inversely proportional to ω^5 for $U_0/C_0 = 0$ and to ω^3 for any U_0/C_0 not equal to zero.

The average Nusselt number for a large artery from $x = 5$ cm. to $x = 20$ cm. was calculated by using Equation (28). (The values of the parameters used in this determination are: $N_{Re} = 200$, $N_{Sc} = 10^3$, $U_0/C_0 = 10^{-2}$, $\omega = 4$, $\lambda = 3$.) The result obtained shows a 7.2% increase in the average Nusselt number over the steady flow value. If the artery was treated as a rigid tube, only a 0.05% increase would have been indicated.

To estimate the range of applicability of this perturbation solution, the ratio $|f_1|/(|M| + |f_2|)$ was calculated for $U_0/C_0 = 10^{-2}$ at the point ($\eta = 2.0$, $\xi = 0.20 \times 10^{-2}$). Results obtained indicate that the perturbation solution is valid over a wider range of the parameters than in the rigid tube case. The range of validity for the rigid tube has been reported elsewhere (*1*).

NOTATION

C = actual wave velocity
C_0 = ideal wave velocity
c = concentration
c_0 = uniform concentration at inlet
c_w = constant concentration at wall
D = diffusion coefficient
E = Young's modulus of elasticity
h = thickness of the wall
N_{Nu} = Nusselt number
N_{Re} = Reynolds number
N_{Sc} = Schmidt number
R = average radius of the conduit
U_0 = time-averaged mean axial velocity
$Re(z)$ = real part of the complex quantity z
$Im(z)$ = imaginary part of the complex quantity z
$Ai(z)$ = airy function

Greek Symbols

α = radial expansion of the tube wall
β = frequency in radians per unit time
σ = Poisson's ratio
ρ = density of the fluid
ρ_w = density of the tube wall
ν = kinematic viscosity
λ = ratio of amplitude of fluctuating component of the pressure gradient to the magnitude of the steady component
ω = dimensionless frequency parameter, $(\beta R^2/\nu)^{1/2}$
$\bar{\omega}$ = dimensionless frequency parameter, $(\beta R^2/D)^{1/2}$
η = dimensionless variable, $y(U_0/RDx)^{1/3}$
ξ = dimensionless x variable, Dx/U_0R^2
ϕ = dimensionless concentration variable,
$\chi(\omega)$ = a function of ω, $[-(8i)/(\omega)^{1/2}[J_1(\omega i^{3/2})]/J_0(\omega i^{3/2})]$
$\Gamma(n)$ = gamma function

LITERATURE CITED

1. Alabastro, E. B. F., and J. D. Hellums, *AIChE J.*, **15**, 164 (1969).
2. Alabastro, E. B. F., Phd. thesis, Rice Univ., Houston, Tex. (1967).
3. McDonald, D. A., "Blood Flow in Arteries," Edward Arnold, Ltd., London, Eng. (1960).
4. Womersley, J. R., *Phys. Med. Biol.*, **2**, 178 (1957).

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